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THE LATTICE OF POSITIVE QUASI-ORDERS ON A SEMIGROUP*

BY

MIROSLAV ĆIRIĆ AND STOJAN BOGDANOVIĆ

Mathematical Institute SANU, Kneza Mihaila 35, 11000 Beograd, Yugoslavia
e-mail: mciric@archimed.filfak.ni.ac.yu stojanb@eknux.eknfak.ni.ac.yu

ABSTRACT

In the present paper we study some properties of positive quasi-orders on semigroups and using these results we describe all semilattice and chain homomorphic images of a semigroup.

Throughout this paper, the notion **poset** will be used as a synonym for the notion "partially ordered set". If ξ is a binary relation on a set A, ξ^{-1} will denote the relation defined by $a\,\xi^{-1}\,b \leftrightarrow b\,\xi\,a$, for $a \in A$, $a\xi = \{x \in A \mid a\,\xi\,x\}$, $\xi a = \{x \in A \mid x\,\xi\,a\}$, for $X \subseteq A$, $X\xi = \bigcup_{x\in X} x\xi$, $\xi X = \bigcup_{x\in X} \xi x$, and the equivalence relations ξ_l and ξ_r on A are defined by: $a\,\xi_l\,b \leftrightarrow a\xi = b\xi$; $a\,\xi_r\,by \leftrightarrow \xi a = \xi b$ $(a, b \in A)$. Let ξ be a relation on a semigroup S. We will say that ξ is **positive** if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$. If for each $a \in S$, $a^2\,\xi\,a$, then ξ is **lower-potent**, and if for any $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$, then ξ satisfies the **cm-property** (common multiple property). By a **quasi-order** we mean a reflexive and transitive binary relation. The poset of quasi-orders on a set A is a complete lattice and it will be denoted by $\mathcal{Q}(A)$. By a **division relation** on a semigroup S we mean a relation $\mid$ defined by: for $a, b \in S$, $a \mid b \leftrightarrow b = xay$ for some $x, y \in S^1$.

A congruence ϱ on a semigroup S is a **semilattice (chain) congruence** if S/ϱ is a semilattice (chain) and then S/ϱ is a **semilattice (chain) homomorphic image** of S. When ϱ is the smallest semilattice congruence on S, S/ϱ will be called a **greatest semilattice homomorphic image** of S.

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An ideal I of a semigroup S is **completely semiprime** if for $a \in S$, $a^2 \in I$ implies $a \in I$, and it is **completely prime** if for $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. By $\mathcal{I}d(S)$ we will denote the lattice of all ideals of a semigroup S and $\mathcal{I}d^{\mathrm{cs}}(S)$ will denote the lattice of all completely semiprime ideals of S. A subset A of a semigroup S is **consistent** if for $a, b \in S$, $ab \in A$ implies $a, b \in A$. A consistent subsemigroup of S will be called a **filter** of S. Clearly, a subset A of a semigroup S is consistent if and only if $S - A$ is an ideal of S, and A is a filter if and only if $S - A$ is a completely prime ideal of S.

For undefined notions and notations we refer to [1], [9] and [16].

Positive quasi-orders on semigroups have been studied from different points of view by B. M. Schein [14], T. Tamura [17–20], M. S. Putcha [10–12] and the others. Their connections with semilattice decompositions of semigroups were investigated the most seriously. T. Tamura [20], by the theorem quoted below as Tamura's theorem, established an isomorphism between the complete lattices of all semilattice congruences of a semigroup and their positive quasi-orders satisfying the cm-property. The smallest elements of these posets, and related greatest semilattice decomposition of a semigroup, were studied systematically in papers of T. Tamura [17–20], M. S. Putcha [11], M. Petrich [8, 9] and by the authors [3, 4]. The greatest semilattice homomorphic image of a semigroup was described by the authors in [4].

In this paper we study yet other aspects of positive quasi-orders of semigroups and using these results we describe *all* semilattice (chain) homomorphic images of a semigroup.

First we will quote the following two results. The first part of Proposition 1 ((i)–(iii)) was proved by G. Birkhoff in [1].

PROPOSITION 1: *Let ξ be a quasi-order of a set A. Then*

(i) $\widetilde{\xi} = \xi \cap \xi^{-1}$ *is an equivalence relation on A;*

(ii) *if E and F are two equivalence classes for $\widetilde{\xi}$, then $x \,\xi\, y$ either for no $x \in E$, $y \in F$ or for all $x \in E$, $y \in F$;*

(iii) *the quotient-set $S/\widetilde{\xi}$ is a poset if $E \leq F$ is defined to mean that $x \,\xi\, y$ for some (hence all) $x \in E$, $y \in F$;*

(iv) *for $a, b \in A$, $a \,\xi\, b$ implies $b\xi \subseteq a\xi$, $\xi a \subseteq \xi b$;*

(v) $\widetilde{\xi} = \xi_l = \xi_r$.

The next theorem is another form of Theorem 3.1 of T. Tamura [20]. As in the proof of Theorem 4.9 of the same paper T. Tamura noticed the lower-potency and compatibility of a positive quasi-order can be replaced by the cm-property.

TAMURA'S THEOREM: *The poset of positive quasi-orders on a semigroup S satisfying the cm-property and the poset of semilattice congruences on S are isomorphic complete lattices. An isomorphism between these lattices is the mapping $\xi \mapsto \widetilde{\xi}$.*

LEMMA 1: *The following conditions for a quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is positive;*
(ii) $(\forall a, b \in S)\ (ab)\xi \subseteq a\xi \cap b\xi$;
(iii) $(\forall a, b \in S)\ \xi a \cup \xi b \subseteq \xi(ab)$;
(iv) *$a\xi$ is an ideal of S, for each $a \in S$;*
(v) *ξa is a consistent subset of S, for each $a \in S$.*

Proof: We will prove only (iii) $\Longrightarrow$ (iv). The rest of the theorem can be proved similarly. Assume $a \in S$, $y \in a\xi$, $x \in S$. Then $a \in \xi y \subseteq \xi(xy)$, so $xy \in a\xi$. Similarly, $yx \in a\xi$. Thus, $a\xi$ is an ideal of S. ∎

A subset K of a lattice L is **closed for meets (joins)** if whenever a subset of K has a meet (join) in L, then this meet (join) lies in K, and it is **closed** if it is closed both for meets and joins. Clearly, any closed subset of a lattice is its sublattice. If L is a lattice with the unity, then any closed sublattice of L containing its unity will be called a **completely closed** sublattice of L. It is easy to verify that the completely closed sublattices of a lattice with the unity forms a complete lattice.

Consider a semigroup S and a completely closed subset K of $\mathcal{I}d(S)$. For any $a \in S$, the family of all elements of K containing a is non-empty and the meet of this set, in notation $K(a)$, lies in K, so $K(a)$ will be called a **principal element** of K generated by a and the set $\{K(a) \mid a \in S\}$ will be called a **principal part** of K. For example, principal elements of $\mathcal{I}d(S)$ are exactly the principal ideals of S. Principal elements of $\mathcal{I}d^{\mathrm{cs}}(S)$, called the principal radicals of S, were described by the authors in [4]. Principal elements of the Boolean sublattice of all 0-consistent ideals of a semigroup with zero, called the principal 0-consistent ideals, have the important role in orthogonal decompositions of semigroups with zero (see [2]).

The following theorem characterizes positive quasi-orders on a semigroup in terms of completely closed sublattices of ideal lattices.

THEOREM 1: *The poset of positive quasi-orders on a semigroup S is a complete lattice and it is dually isomorphic to the lattice of completely closed sublattices of $\mathcal{I}d(S)$.*

Proof: As M. S. Putcha [10] mentioned, a quasi-order on a semigroup S is positive if and only if it contains the division relation on S, whence the set of positive quasi-orders on S is a principal dual ideal of $\mathcal{Q}(S)$ generated by the division relation on S, and hence it is a complete lattice.

Let ξ be a positive quasi-order on S and let

$$K_\xi = \{I \in \mathcal{I}d(S) \mid I\xi = I\}. \tag{1}$$

Clearly, $a\xi \in K_\xi$, for any $a \in S$, $S \in K_\xi$ and K_ξ is closed for joins. Further, let $\{I_\alpha \mid \alpha \in Y\}$ be a subset of K_ξ having a meet I in $\mathcal{I}d(S)$. For $a \in I$ we have that $a \in I_\alpha$, whence $a\xi \subseteq I_\alpha\xi = I_\alpha$, for any $\alpha \in Y$, so $a\xi \subseteq I$. Thus, $I\xi = \bigcup_{a\in I} a\xi \subseteq I$, whence $I\xi = I$ and $I \in K_\xi$. Hence, K_ξ is closed for meets, so it is a completely closed sublattice of $\mathcal{I}d(S)$. Also, it is clear that $K_\xi(a) = a\xi$, for any $a \in S$.

Let K be a completely closed sublattice of $\mathcal{I}d(S)$. Define a relation ξ on S by:

$$a\,\xi\,b \quad \leftrightarrow \quad K(b) \subseteq K(a) \quad (a, b \in S). \tag{2}$$

It is easy to verify that ξ is a quasi-order on S and that $a\xi = K(a)$, for any $a \in S$, so by Lemma 1, ξ is positive. If $I \in K$ and $x \in I\xi$, then $a\,\xi\,x$, for some $a \in I$, so $x \in K(x) \subseteq K(a) \subseteq I$, whence $I\xi \subseteq I$, i.e. $I\xi = I$, and therefore, $K \subseteq K_\xi$. Conversely, if $I \in K_\xi$, then $I = I\xi = \bigcup_{a\in I} a\xi = \bigcup_{a\in I} K(a) \in K$, since K is closed for joins, so $K_\xi \subseteq K$. Therefore, $K = K_\xi$, so the mapping $\xi \mapsto K_\xi$ maps the lattice of positive quasi-orders on S onto the lattice of completely closed sublattices of $\mathcal{I}d(S)$.

Further, consider positive quasi-orders ξ and η on S. If $\xi \subseteq \eta$, then by $I \in K_\eta$ it follows that $I\xi = \bigcup_{a\in I} a\xi \subseteq \bigcup_{a\in I} a\eta = I\eta = I$, whence $I = I\xi$ and hence $I \in K_\xi$. Conversely, by $K_\eta \subseteq K_\xi$ it follows that $a\xi = K_\xi(a) = K_\eta(a) = a\eta$, for each $a \in S$, whence $\xi \subseteq \eta$. Therefore, $\xi \subseteq \eta$ if and only if $K_\eta \subseteq K_\xi$, whence the mapping $\xi \mapsto K_\xi$ is a dual order-isomorphism, so by the dual of Lemma II 3.2 [1], it is a dual lattice-isomorphism. ∎

THEOREM 2: *The poset of lower-potent positive quasi-orders on a semigroup S is a complete lattice and it is dually isomorphic to the lattice of completely closed sublattices of $\mathcal{I}d^{cs}(S)$.*

Proof: As T. Tamura [18] proved, the poset of lower-potent positive quasi-orders on S has a smallest element μ, by Theorem 1, any quasi-order on S containing μ is positive, since it also contains the division relation on S, and clearly, it is lower-potent. Therefore, the set of lower-potent positive quasi-orders on S is a principal dual ideal of $\mathcal{Q}(S)$ generated by μ, and hence, it is a complete lattice.

Let ξ be a positive quasi-order on S. To establish an isomorphism between the lattice of lower-potent positive quasi-orders on S and the lattice of completely closed sublattices of $\mathcal{I}d^{cs}(S)$ it is sufficient to prove that ξ is lower-potent if and only if $K_\xi \subseteq \mathcal{I}d^{cs}(S)$. Let ξ be lower-potent. Then for $a, x \in S$, by $x^2 \in a\xi$ it follows that $a\,\xi\,x^2\,\xi\,x$, so $a\,\xi\,x$, i.e. $x \in a\xi$. Therefore, $a\xi$ is completely semiprime, for each $a \in S$, whence $K_\xi \subseteq \mathcal{I}d^{cs}(S)$. Conversely, if $K_\xi \subseteq \mathcal{I}d^{cs}(S)$, then $a^2\xi$ is completely semiprime, whence $a \in a^2\xi$, i.e. $a^2\,\xi\,a$, for each $a \in S$, so ξ is lower-potent. ∎

LEMMA 2: *The following conditions for a quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is positive and satisfies the cm-property;*
(ii) *ξa is a filter of S, for each $a \in S$;*
(iii) *$(\forall a, b \in S)\ a\xi \cap b\xi = (ab)\xi$.*

Proof: (i) $\Longrightarrow$ (ii). For $a \in S$, ξa is a subsemigroup of S, so by Lemma 1, ξa is a filter of S.

(ii) $\Longrightarrow$ (iii). Assume $a, b \in S$. If $x \in a\xi \cap b\xi$ then $a, b \in \xi x$, so $ab \in \xi x$, i.e. $x \in (ab)\xi$, since ξx is a subsemigroup of S. Conversely, if $x \in (ab)\xi$ then $ab \in \xi x$, whence $a, b \in \xi x$, i.e. $x \in a\xi \cap b\xi$, since ξx is consistent. Hence, (iii) holds.

(iii) $\Longrightarrow$ (i). By Lemma 1, ξ is positive, and clearly, ξ satisfies the cm-property. ∎

A subset A of a lattice L is **meet-dense** in L if any element of L can be represented as a meet of some subset of A. We will say that a sublattice K of $\mathcal{I}d^{cs}(S)$ satisfies the **cpi-property** (completely prime ideal-property) if the set of all completely prime ideals of S that are elements of K is meet-dense in K. Note that this property is very significant in theories of semigroups, rings and lattices.

For semigroups, this condition was proved for the lattice $\mathcal{I}d^{cs}(S)$ in some special cases by Š. Schwarz [15] and K. Iséki [5], and in the general case by M. Petrich [9]. The same result, without use of Zorn's lemma arguments, was proved by the authors in [4]. The related result in the theory of lattices is known as the prime ideal theorem, and for the related results for rings we refer to W. Krull [6] and N. H. McCoy [7]. By the following theorem we establish a connection between the cm-property for positive quasi-orders and the cpi-property for related completely closed sublattices of $\mathcal{I}d^{cs}(S)$.

THEOREM 3: *The poset of positive quasi-orders on a semigroup S satisfying the cm-property and the poset of completely closed sublattices of $\mathcal{I}d^{cs}(S)$ satisfying the cpi-property are dually isomorphic complete lattices.*

Proof: Let P denote the poset of positive quasi-orders on S satisfying the cm-property and let P' denote the poset of completely closed sublattices of $\mathcal{I}d^{cs}(S)$ satisfying the cpi-property. Consider a mapping $\xi \mapsto K_\xi$, $\xi \in P$, defined as in Theorem 1. To prove that this mapping is a dual order-isomorphism of P onto P', it is sufficient to prove that a positive quasi-order ξ on S satisfies the cm-property if and only if K_ξ satisfies the cpi-property. In order to simplify our notations, let $K = K_\xi$.

Let ξ satisfy the cm-property. Assume $a \in S$. By Lemma 2, $S - \xi a$ is a completely prime ideal of S. Assume $x \in S - \xi a$, $y \in x\xi$. If $y \notin S - \xi a$, i.e. $y \in \xi a$, then $y \,\xi\, a$, so by $x \,\xi\, y$ we obtain $x \,\xi\, a$, i.e. $x \in \xi a$, which contradicts our starting hypothesis. Hence $y \in S - \xi a$, so $x\xi \subseteq S - \xi a$, for each $x \in S - \xi a$. Thus $S - \xi a = \bigcup_{x \in S - \xi a} x\xi$. Therefore, $S - \xi a \in K$, for each $a \in S$.

Assume an arbitrary $I \in K$. Let us prove that $I = \bigcap_{a \in S - I}(S - \xi a)$. If $x \in I$ and $x \notin S - \xi a$, i.e. $x \in \xi a$, for some $a \in S - I$, then $a \in x\xi \subseteq I$, which is in contradiction with $a \in S - I$. Therefore, $I \subseteq \bigcap_{a \in S - I}(S - \xi a)$. On the other hand, if $x \in \bigcap_{a \in S - I}(S - \xi a)$ and $x \notin I$, then $x \in S - I$, whence $x \in S - \xi x$, i.e. $x \notin \xi x$, which is not true. Thus, $\bigcap_{a \in S - I}(S - \xi a) \subseteq I$. Hence, I is the intersection of a family of completely prime ideals from K, so K satisfies the cpi-property.

Conversely, let K satisfy the cpi-property. Assume $a, b \in S$. Then $K(ab) = \bigcap_{\alpha \in Y} P_\alpha$, where P_α, $\alpha \in Y$, are completely prime ideals of S and elements of K. Let $U = \{\alpha \in Y \mid a \in P_\alpha\}$, $V = \{\beta \in Y \mid b \in P_\beta\}$. For each $\alpha \in Y$, $ab \in P_\alpha$, whence $a \in P_\alpha$ or $b \in P_\alpha$, since P_α is completely prime, so $Y = U \cup V$. Also, without loss of generality we can assume that $U \neq \emptyset$ and $V \neq \emptyset$ (for example, we can assume that S is one of P_α). For each $\alpha \in U$, $a \in P_\alpha$ implies $K(a) \subseteq P_\alpha$, since $P_\alpha \in K$, so $K(a) \subseteq \bigcap_{\alpha \in U} P_\alpha$. Similarly, $K(b) \subseteq \bigcap_{\beta \in V} P_\beta$. Hence

$$K(a) \cap K(b) \subseteq \left(\bigcap_{\alpha \in U} P_\alpha\right) \bigcap \left(\bigcap_{\beta \in V} P_\beta\right) = \bigcap_{\alpha \in Y} P_\alpha = K(ab),$$

so by (2) we obtain that ξ satisfies the cm-property.

Therefore, P and P' are dually isomorphic posets, and P is a complete lattice, so P' is also a complete lattice. ∎

By Lemma 2, the principal part of the lattice K_ξ corresponding to a positive quasi-order ξ on a semigroup S is a meet-subsemilattice of K_ξ and it is a homomorphic image of S. Now we go to the main theorem

THEOREM 4: *A semilattice Y is a semilattice homomorphic image of a semigroup S if and only if it is isomorphic to the principal part of some completely closed sublattice of $\mathcal{I}d^{cs}(S)$ satisfying the cpi-property.*

Especially, the principal part of $\mathcal{I}d^{cs}(S)$ is the greatest semilattice homomorphic image of S.

Proof: This follows by Tamura's theorem and by Theorem 3. ∎

Also, we will characterize quasi-orders that produce decompositions into a chain of semigroups.

LEMMA 3: *The following conditions for a quasi-order ξ of a semigroup S are equivalent:*

(i) *ξ is positive, linear and satisfies the cm-property;*
(ii) *ξ is positive and for all $a, b \in S$, $ab\,\xi\,a$ or $ab\,\xi\,b$;*
(iii) *$a\xi$ is a completely prime ideal of S, for each $a \in S$;*
(iv) *$(\forall a, b \in S)\ \xi a \cup \xi b = \xi(ab)$.*

Proof: (i) $\Longrightarrow$ (ii). Assume $a, b \in S$. By (i), $a\,\xi\,b$ or $b\,\xi\,a$. Assume that $a\,\xi\,b$. Then $a, b \in \xi b$, so by Lemma 2, $ab \in \xi b$, i.e. $ab\,\xi\,b$. Similarly we prove that $b\,\xi\,a$ implies $ab\,\xi\,a$.

(ii) $\Longrightarrow$ (iii). Assume $a \in S$. By Lemma 1, $a\xi$ is an ideal of S. Assume $x, y \in S$ such that $xy \in a\xi$, i.e. $a\,\xi\,xy$. Then by (ii), $a\,\xi\,x$ or $a\,\xi\,y$, i.e. $x \in a\xi$ or $y \in a\xi$. Thus, (iii) holds.

(iii) $\Longrightarrow$ (iv). Assume $a, b \in S$. By Lemma 1, $\xi a \cup \xi b \subseteq \xi(ab)$. Assume $x \in \xi(ab)$. Then $x \,\xi\, ab$, i.e. $ab \in x\xi$, so by (iii), $a \in x\xi$ or $b \in x\xi$, i.e. $x \in \xi a$ or $x \in \xi b$. Thus, $\xi(ab) \subseteq \xi a \cup \xi b$.

(iv) $\Longrightarrow$ (i). By Lemma 1, ξ is positive. Assume $a \in S$, $x, y \in \xi a$. Then $xy \in \xi(xy) = \xi x \cup \xi y \subseteq \xi a$, by (iv) and by Proposition 1. Therefore, ξa is a subsemigroup of S, for each $a \in S$, so by Lemmas 2 and 4 we obtain that ξ is positive and ξ satisfies the cm-property. Finally, for $a, b \in S$, by (iv), $ab \in \xi a$ or $ab \in \xi b$. Since ξa and ξb are filters, then $b \in \xi a$ or $a \in \xi b$, so ξ is linear. ∎

Using Lemma 3 and Tamura's theorem, we obtain the following

PROPOSITION 2: *The poset of positive, linear quasi-order of a semigroup S satisfying the cm-property is isomorphic to the poset of chain congruences on S.*

LEMMA 4: *Let I and J be two ideals of a semigroup S such that $I \cap J$ is completely prime. Then $I \subseteq J$ or $J \subseteq I$.*

Proof: Suppose that there exists $a \in I - J$, $b \in J - I$. Since I and J are ideals, $ab \in I \cap J$, whence $a \in I \cap J$ or $b \in I \cap J$, since $I \cap J$ is completely prime. Thus, we obtain that $a \in J$ or $b \in I$, which contradicts our starting hypothesis. Hence, $I \subseteq J$ or $J \subseteq I$. ∎

The following theorem characterizes positive, linear quasi-orders satisfying the cm-property in terms of completely prime ideals.

THEOREM 5: *The following conditions for a positive quasi-order ξ on a semigroup S are equivalent:*

(i) *ξ is linear and satisfies the cm-property;*
(ii) *ξ satisfies the cm-property and the poset of all completely prime ideals from K_ξ is a chain;*
(iii) *K_ξ consists of completely prime ideals.*

Proof: (i) $\Longrightarrow$ (iii). This follows by Lemma 3 and by the definition of K_ξ.

(iii) $\Longrightarrow$ (ii). This follows by Theorem 3 and by Lemma 4.

(ii) $\Longrightarrow$ (i). Assume $a, b \in S$. By Lemma 2, ξa and ξb are filters, i.e. $I = S - \xi a$ and $J = S - \xi b$ are completely prime ideals of S. By the proof of Theorem 3, $I, J \in K_\xi$, so by (ii), $I \subseteq J$ or $J \subseteq I$. Thus, $\xi b \subseteq \xi a$ or $\xi a \subseteq \xi b$, so ξ is linear.
∎

THEOREM 6: *A chain Y is a chain homomorphic image of a semigroup S if and only if it is isomorphic to the principal part of some completely closed sublattice of $\mathcal{I}d^{cs}(S)$ consisting of completely prime ideals.*

Proof: This follows by Theorem 4, Proposition 2 and by Theorem 5. ∎

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